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ASYMPTOTIC ANALYSIS OF AN END-LOADED, TRANSVERSELY ISOTROPIC, ELASTIC, SEMI-INFINITE STRIP WEAK IN SHEAR

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Abstract—We use linear elasticity to study a transversely isotropic (or specially orthotropic), semi-infinite slab in plane strain, free of traction on its faces and at infinity and subject to edge loads or displacements that produce stresses and displacements that decay in the axial direction. The governing equations (which are identical to those for a strip in plane stress, free of traction on its long sides and at infinity, and subject to tractions or displacements on its short side) are reduced, in the standard way, to a fourth-order partial differential equation with boundary conditions for a dimensionless Airy stress function, f. We study the asymptotic solutions to this equation for four sets of end conditions—traction, mixed (two), displacement—as ε, the ratio of the shear modulus to the geometric mean of the axial and transverse extensional moduli, approaches zero. In all cases, the solutions for f consist of a "wide" boundary layer that decays slowly in the axial direction (over a distance that is long compared to the width of the strip) plus a "narrow" boundary layer that decays rapidly in the axial direction (over a distance that is short compared to the width of the strip). Moreover, we find that the narrow boundary layer has a "sinuous" part that varies rapidly in the transverse direction, but which, to lowest order, does not enter the boundary conditions nor affect the transverse normal stress or the displacements. Because the exact biorthogonality condition for the eigenfunctions associated with f can be replaced by simpler orthogonality conditions in the limit as $\varepsilon \to 0$, we are able to obtain, to lowest order, explicit formulae for the coefficients in the eigenfunction expansions of f for the four different end conditions.

1. INTRODUCTION

The stress analysis of semi-infinite, transversely isotropic (or specially orthotropic) linearly elastic strips† subject to end loads or displacements not only illuminates the dependence of Saint-Venant's Principle on the elastic constants of a material, but also arises in the consideration of end effects in attempts to improve elementary beam theory. In particular, if the beam is weak in shear as in a strongly anisotropic composite, then such end effects may dominate. The problem is also interesting mathematically because it shows, once again, that if an equation contains a parameter, ε, then (after various preliminary scalings of the variables) the limiting forms of the equation as $\varepsilon \to 0$ may differ considerably from their parent.

For an elastically *isotropic* strip, the equations of linear plane strain (or plane stress) theory may be reduced to a *biharmonic* equation for a dimensionless Airy stress function, f. In this case, neither the differential equation nor the boundary conditions contain a small parameter. As Gregory and Gladwell (1982) show, it takes considerable analysis to get useful solutions. However, as we shall show, it is easier to get solutions for a strip weak in shear because the governing equations now contain a small parameter, ε, that measures, essentially, the ratio of the shear to the geometric mean of the extensional moduli. If we scale the independent variables properly (by fractional powers of ε), then solutions of the elliptic, fourth-order partial differential equation for f and the associated biorthogonality conditions simplify as $\varepsilon \to 0$, allowing us to obtain, at least *formally* and to lowest order, explicit solutions for various end data. If ε is not small, these end data, if they are to imply decaying solutions, cannot be totally arbitrary but must satisfy certain obvious and not so obvious integral conditions, as explained by Gregory and Wan (1984) and Lin and Wan (1988). However, as $\varepsilon \to 0$, we shall show that these integral conditions simplify considerably. In the future, we hope to show how these solutions may be used to improve the lower

† That is, strips whose axes of elastic orthotropy are parallel and normal to the sides of the strips.

natural frequencies of vibration of a cantilevered beam predicted by elementary theory, using recent work of Duva and Simmonds (1991).

We note that asymptotic analyses of strongly anisotropic materials have been considered by other authors for fiber-reinforced composites. For example, Everstine and Pipkin (1973) have investigated boundary-layer phenomena for end-loaded, fiber-reinforced cantilevered beams. See also Everstine and Pipkin (1971), Spencer (1974) and the review article by Pipkin (1979) for further details.

2. THE GOVERNING EQUATIONS

We consider a slab which, before deformation, occupies the region $x \geqslant 0$, $|y| \leqslant H$ in a fixed Cartesian reference frame, $Oxyz$. In the linear theory of plane strain, the stresses parallel to the xy-plane, σ_x, τ and σ_y, are functions of x and y only and satisfy the equilibrium equations

$$\sigma_{x,x}+\tau_{,y}=0, \quad \tau_{,x}+\sigma_{y,y}=0, \tag{1}$$

where a comma denotes partial differentiation with respect to the subscript that follows.

If U and V are displacements in the x- and y-directions, respectively, then the strain–displacement relations are

$$e_x=U_{,x}, \quad \gamma=U_{,y}+V_{,x}, \quad e_y=V_{,y} \tag{2}$$

and the strain–stress relations for a material transversely isotropic about the x-axis may be given the form

$$E_x e_x=\sigma_x-\nu(E_x/E_y)\sigma_y, \quad G\gamma=\tau, \quad E_y e_y=\sigma_y-\nu\sigma_x. \tag{3}$$

Mathematically, (1)–(3) are identical (save for different elastic constants) to those relations for a transversely isotropic elastic strip in a state of plane stress.

The strains must satisfy the compatibility condition

$$e_{y,xx}-\gamma_{,xy}+e_{x,yy}=0 \tag{4}$$

and the equilibrium eqns (1), may be satisfied identically by setting

$$\sigma_x=F_{,yy}, \quad \tau=-F_{,xy}, \quad \sigma_y=F_{,xx}, \tag{5}$$

where F is the Airy stress function. Substituting (5) into (3) and the resulting equations into (4), we obtain our basic field equation

$$F_{,xxxx}+(E_y/G-2\nu)F_{,xxyy}+(E_y/E_x)F_{,yyyy}=0, \quad x>0, \quad |y|<H. \tag{6}$$

In the following, we shall assume that F is C^4 on the interior of the strip.

The long sides of the strip are traction free; hence, by (5),

$$F_{,xx}(x,\pm H)=F_{,xy}(x,\pm H)=0. \tag{7}$$

We also shall assume that the stresses decay to zero as $x\to\infty$, uniformly in y. This implies that the loads on any section $x=$ constant are self-equilibrating, i.e. the net force and moment vanish. Since we may always add a linear function of x and y to F without affecting the stresses, we may, without loss of generality, integrate the boundary conditions (7) with respect to x and discard the constants of integration to obtain

$$F(x, \pm H) = F_{,y}(x, \pm H) = 0. \tag{8}$$

Finally, we shall assume that $F_{,x}$ and $F_{,y}$ are continuous on the boundary of the strip.

On the left end of the strip, we consider four types of boundary conditions. Using the classification of Gregory and Wan (1984), we have, for $|y| < H$,

$$\text{Case A:} \quad \sigma_x(0, y) = \hat{\sigma}_x(y), \quad \tau(0, y) = \hat{\tau}(y) \tag{9A}$$

$$\text{Case B:} \quad \sigma_x(0, y) = \hat{\sigma}_x(y), \quad V(0, y) = \hat{V}(y) \tag{9B}$$

$$\text{Case C:} \quad U(0, y) = \hat{U}(y), \quad \tau(0, y) = \hat{\tau}(y) \tag{9C}$$

$$\text{Case D:} \quad U(0, y) = \hat{U}(y), \quad V(0, y) = \hat{V}(y), \tag{9D}$$

where a hat (^) denotes a prescribed function.

To convert (9A) into boundary conditions on F, we first integrate the relation $F_{,yy}(0, y) = \hat{\sigma}_x(y)$ and $F_{,xy}(0, y) = -\hat{\tau}(y)$ with respect to y. Noting that $\hat{\sigma}_x(y)$ and $\hat{\tau}(y)$ are self-equilibrating and that the first partial derivatives of F, by hypothesis, are continuous on the boundary of the strip, we may replace (9A) by

$$F(0, y) = \hat{F}(y) \equiv \int_{-H}^{y} (y-t)\hat{\sigma}_x(t)\,\mathrm{d}t, \quad F_{,x}(0, y) = \hat{F}_x(y) \equiv -\int_{-H}^{y} \hat{\tau}(t)\,\mathrm{d}t. \tag{10A}$$

To convert (9B), we assume that $\hat{V}(y)$ is differentiable. It then follows from (2), (3), (5) and (10A) that

$$F(0, y) = \hat{F}(y), \quad F_{,xx}(0, y) = [E_y\hat{V}(y) + v\hat{F}'(y)]'. \tag{10B}$$

To convert (9C), we assume that $\hat{U}(y)$ is twice differentiable. It then follows from (2), (3), (5) and (10A) that

$$F_{,xxx}(0, y) = -[E_y\hat{U}(y) + (E_y/G - v)\hat{F}_x(y)]'', \quad F_{,x}(0, y) = \hat{F}_x(y). \tag{10C}$$

The conversion of (9D) follows from (10C) and (10B) as

$$F_{,xxx}(0, y) + (E_y/G - v)F_{,xyy}(0, y) = -E_y\hat{U}''(y), \quad F_{,xx}(0, y) - vF_{,yy}(0, y) = E_y\hat{V}'(y). \tag{10D}$$

The axial and transverse displacements, U and V, may be computed in terms of F by first using (2), (3) and (5) to obtain

$$E_x U_{,x} = F_{,yy} - v(E_x/E_y)F_{,xx} \tag{11}$$

$$GV_{,xx} = -[(G/E_x)F_{,yyy} + (1 - vG/E_y)F_{,xxy}]. \tag{12}$$

If we assume that U and V, as well as F, decay exponentially as $x \to \infty$, it follows upon integrating (11) once and (12) twice from x to ∞ that

$$E_x U = -\int_x^{\infty} F_{,yy}(t, y)\,\mathrm{d}t - v(E_x/E_y)F_{,x}(x, y) \tag{13}$$

and

$$GV = (G/E_x)\int_x^{\ell} (x-t)F_{,yyy}(t,y)\,\mathrm{d}t - (1-\nu G/E_y)F_{,y}(x,y). \tag{14}$$

3. NON-DIMENSIONALIZATION

Let

$$x = (E_x/E_y)^{1/4}H\xi, \quad y = H\eta, \quad F = \sigma_0 H^2 f(\xi,\eta), \tag{15}$$

where σ_0 is a constant reference stress.

To concentrate on the effects of weak resistance to shearing, we now set

$$(E_x/E_y)^{1/2}(E_y/G-2\nu) = \varepsilon^{-1}, \quad \tilde{\nu} = \nu(E_x/E_y)^{1/2}. \tag{16}$$

We assume henceforth that $\varepsilon > 0$. Our primary objective is to ascertain the asymptotic character of the stresses and displacements as $\varepsilon \to 0$. Further, to guarantee that we may choose the reference stress, σ_0, so that $f = O(1)$ everywhere as $\varepsilon \to 0$, we assume that $\tilde{\nu} = O(1)$ and non-dimensionalize the displacements and the edge data, $\hat{F}$ and $\hat{F}_x$, of Cases A and C as follows:

$$U = (\sigma_0 H/E_x^{3/4}E_y^{1/4})\varepsilon^{-1/2}u(\xi,\eta), \quad V = (\sigma_0 H/E_x^{1/2}E_y^{1/2})[\varepsilon^{-1}\kappa(\xi)+v(\xi,\eta)]$$

$$\hat{F} = \sigma_0 H^2\hat{f}(\eta), \quad \hat{F}_x = \varepsilon^{1/2}(E_y/E_x)^{1/4}\sigma_0 H\hat{f}_\xi(\eta). \tag{17†}$$

Thus, our basic partial differential equation, (6), the traction-free conditions on the sides of the strip, (8), and the various stress/displacement conditions on the left end, (10A)–(10D), take the dimensionless forms

$$f_{,\xi\xi\xi\xi} + \varepsilon^{-1}f_{,\xi\xi\eta\eta} + f_{,\eta\eta\eta\eta} = 0, \quad \xi > 0, \quad |\eta| < 1 \tag{18}$$

$$f(\xi,\pm 1) = f_{,\eta}(\xi,\pm 1) = 0 \tag{19}$$

$$f(0,\eta) = \hat{f}(\eta), \quad f_{,\xi}(0,\eta) = \varepsilon^{1/2}\hat{f}_\xi(\eta) \tag{20A}$$

$$f(0,\eta) = \hat{f}(\eta), \quad f_{,\xi\xi}(0,\eta) = [\hat{v}(\eta)+\tilde{\nu}\hat{f}'(\eta)]' \tag{20B}$$

$$f_{,\xi\xi\xi}(0,\eta) = -\varepsilon^{-1/2}[\hat{u}(\eta)+(1+\tilde{\nu}\varepsilon)\hat{f}_\xi(\eta)]'', \quad f_{,\xi}(0,\eta) = \varepsilon^{1/2}\hat{f}_\xi(\eta) \tag{20C}$$

$$f_{,\xi\xi\xi}(0,\eta) + (\varepsilon^{-1}+\tilde{\nu})f_{,\xi\eta\eta}(0,\eta) = -\varepsilon^{-1/2}\hat{u}''(\eta), \quad f_{,\xi\xi}(0,\eta) - \tilde{\nu}f_{,\eta\eta}(0,\eta) = \hat{v}'(\eta). \tag{20D}$$

We observe that the fourth-order partial differential eqn (18) is elliptic provided that $\varepsilon > 0$. However, as $\varepsilon \to 0$, we encounter a singular perturbation problem, because formally setting $\varepsilon = 0$ in (18) yields a hyperbolic equation. The expected difficulty in satisfying all of the boundary conditions will become apparent later [cf. the related discussion in Everstine and Pipkin (1971) and Horgan (1982)].

From (13), (14) and (17), we obtain the following expressions for the dimensionless axial and transverse displacements:

$$\varepsilon^{-1/2}u = -\int_{\xi}^{L} f_{,\eta\eta}(\zeta,\eta)\,\mathrm{d}\zeta - \tilde{\nu}f_{,\xi}(\xi,\eta) \tag{21}$$

and

† Without a further condition—which we shall specify presently in (25) where there is motivation—we cannot uniquely determine κ and v given V.

$$\kappa+\varepsilon v = \varepsilon\int_{\xi}^{\infty}(\xi-\zeta)f_{,\eta\eta\eta}(\zeta,\eta)\,\mathrm{d}\zeta-(1+\bar{v}\varepsilon)f_{,\eta}(\xi,\eta). \tag{22}$$

We may obtain three useful identities involving u, κ, v and derivatives of f only at any value of ξ by multiplying (21) by 1 or η and (22) by $1-\eta^2$ and integrating with respect to η from -1 to 1. Integrating by parts and using the traction-free boundary conditions, (19), we obtain

$$\int_{-1}^{1}\{1,\eta\}[u(\xi,\eta)+\bar{v}\varepsilon^{1/2}f_{,\xi}(\xi,\eta)]\,\mathrm{d}\eta = 0,\quad \xi\geqslant 0 \tag{23}$$

and

$$(4/3)\kappa(\xi) = -\int_{-1}^{1}[\varepsilon(1-\eta^2)v(\xi,\eta)+2(1+\bar{v}\varepsilon)\eta f(\xi,\eta)]\,\mathrm{d}\eta,\quad \xi\geqslant 0. \tag{24}$$

To determine κ and v uniquely in $(17)_2$, given V, we shall require that

$$\int_{-1}^{1}(1-\eta^2)v(\xi,\eta)\,\mathrm{d}\eta = 0,\quad \xi\geqslant 0. \tag{25}$$

Thus, (24) reduces to

$$(2/3)\kappa(\xi) = -(1+\bar{v}\varepsilon)\int_{-1}^{1}\eta f(\xi,\eta)\,\mathrm{d}\eta,\quad \xi\geqslant 0. \tag{26}$$

We may now derive an expression for v alone, in terms of f, as follows. First, we use (2), (5), (15) and (16) to write the strain–stress relation $(3)_3$ in the dimensionless form

$$v_{,\eta} = f_{,\xi\xi}-\bar{v}f_{,\eta\eta}. \tag{27}$$

Then, integrating both sides from -1 to η and using (25) to solve for $v(\xi,-1)$, we obtain

$$v = \int_{-1}^{1}\{[(\eta/4)(3-\eta^2)-1/2]f_{,\xi\xi}(\xi,\eta)+(3\bar{v}/2)\eta f(\xi,\eta)\}\,\mathrm{d}\eta+\int_{-1}^{\eta}f_{,\xi\xi}(\xi,\mu)\,\mathrm{d}\mu-\bar{v}f_{,\eta}(\xi,\eta). \tag{28}$$

4. NECESSARY CONDITIONS FOR DECAY

Gregory and Wan (1984) and Lin and Wan (1988) [see also Lin and Wan (1990)] have derived, in different ways, necessary conditions for the edge data in Cases A–D to be compatible with exponential decay of u, κ, v and f as $\xi\to\infty$.

Overall equilibrium requires that the edge data produce no stress resultants or couple. From (10A) and (17) this implies that

$$\hat{f}(\pm 1) = \hat{f}'(\pm 1) = \hat{f}_{,\xi}(\pm 1) = 0, \tag{29}$$

i.e. whenever f or $f_{,\xi}$ are prescribed on the ends of the strip, these data must be compatible with the traction-free boundary conditions (19).

To find additional necessary conditions on the edge data in Cases C and B, we set $\xi = 0$ in (23) and (26), and, noting $(20\text{C})_2$, get

$$\int_{-1}^{1} \{1, \eta\}[\hat{u}(\eta) + \bar{\nu}\varepsilon \hat{f}_{;\xi}(\eta)]\, d\eta = 0 \tag{30}$$

and

$$(2/3)\hat{\kappa}(0) = -(1+\bar{\nu}\varepsilon)\int_{-1}^{1} \eta \hat{f}(\eta)\, d\eta. \tag{31}$$

These are special cases, respectively, of eqns (5.8a, b) and (5.9) in Lin and Wan (1988).

To obtain necessary conditions for Case D, we follow Gregory and Wan (1984) and introduce the dimensionless stress functions g^i, $i = \{\mathrm{T, B, F}\}$, where

$$g^i \sim g^i_{\mathrm{S}} \equiv \{(1/2)\eta^2, (1/6)\eta^3, -(1/6)\varepsilon^{1/2}\xi\eta(\eta^2-3)\} \quad \text{as} \quad \xi \to \infty, \tag{32}$$

T, B and F being mnemonics for tension, bending and flexure, and the subscript "S" denoting the Saint-Venant solution for a strip. The associated displacements, with rigid body terms chosen so that

$$\int_{-1}^{1} \{1, \eta\} u^i_{\mathrm{S}}(0, \eta)\, d\eta = \int_{-1}^{1} (1-\eta^2) v^i_{\mathrm{S}}(\xi, \eta)\, d\eta = 0,$$

are

$$u^i_{\mathrm{S}} = \{\varepsilon^{1/2}\xi, \varepsilon^{1/2}\xi\eta, -(1/2)\varepsilon\xi^2\eta + (1/6)(1+\bar{\nu}\varepsilon)\eta(\eta^2-3/5)\} \tag{33}$$

$$\begin{aligned} \kappa^i_{\mathrm{S}} &= \{0, -(1/2)\varepsilon\xi^2, -(1/30)\varepsilon^{1/2}\xi[12-\varepsilon(5\xi^2-24\bar{\nu})]\} \\ v^i_{\mathrm{S}} &= \{-\bar{\nu}\eta, -(\bar{\nu}/2)(\eta^2-1/5), (\bar{\nu}/2)\varepsilon^{1/2}\xi(\eta^2-1/5)\}. \end{aligned} \tag{34}$$

[Because the strip is weak in shear, the factor of $\varepsilon^{1/2}$ in the third element on the right of (32) is necessary if the displacements in (33) and (34) are to be O(1) at $\xi = 0$.]

In addition to the basic field equation, (18), g^i must satisfy the traction-free conditions

$$g^i_{;\xi\xi}(\xi, \pm 1) = g^i_{;\xi\eta}(\xi, \pm 1) = 0 \tag{35}$$

and the zero-displacement conditions

$$g^i_{;\xi\xi\xi}(0, \eta) + (\varepsilon^{-1} + \bar{\nu}) g^i_{;\xi\eta\eta}(0, \eta) = g^i_{;\xi\xi}(0, \eta) - \bar{\nu} g^i_{;\eta\eta}(0, \eta) = 0 \tag{36}$$

which follow from (20D). The Betti Reciprocity Principle then implies that

$$\int_{-1}^{1} [\hat{u}(\eta) g^i_{;\eta\eta}(0, \eta) - \varepsilon^{1/2}\hat{v}(\eta) g^i_{;\xi\eta}(0, \eta)]\, d\eta = \{0, 0, (2/3)\hat{\kappa}(0)\}. \tag{37}$$

Setting

$$g^i = \bar{g}^i(\xi, \eta; \varepsilon) + g^i_{\mathrm{S}}(\xi, \eta) \tag{38}$$

and noting (25) and (32), we may rewrite (37) as

$$\int_{-1}^{1} [\hat{u}(\eta)\{1, \eta, 0\} + \hat{u}''(\eta)\bar{g}^i(0, \eta) + \varepsilon^{1/2}\hat{v}'(\eta)\bar{g}^i_{;\xi}(0, \eta)]\, d\eta = \{0, 0, (2/3)\hat{\kappa}(0)\}. \tag{39}$$

Thus, to use (39) to find necessary conditions for decay in case D, we must find $\bar{g}^i$. That is, the following three *canonical problems* must be solved:

$$\tilde{g}^i_{,\xi\xi\xi\xi}+\varepsilon^{-1}\tilde{g}^i_{,\xi\xi\eta\eta}+\tilde{g}^i_{,\eta\eta\eta\eta} = 0, \quad \xi > 0, \quad |\eta| < 1 \tag{40}$$

$$\tilde{g}^i(\xi, \pm 1) = \tilde{g}^i_{,\eta}(\xi, \pm 1) = 0, \tag{41}$$

$$\tilde{g}^i_{,\xi\xi\xi}(0,\eta)+(\varepsilon^{-1}+\tilde{\nu})\tilde{g}^i_{,\xi\eta\eta}(0,\eta) = \varepsilon^{-1/2}\{0,0,(1+\tilde{\nu}\varepsilon)\eta\}, \quad \tilde{g}^i_{,\xi\xi}(0,\eta)-\tilde{\nu}\tilde{g}^i_{,\eta\eta}(0,\eta) = \tilde{\nu}\{1,\eta,0\}, \tag{42}$$

$$\tilde{g}^i, \tilde{g}^i_{,\xi} \to 0 \quad \text{as } \xi \to \infty, \quad \text{uniformly in } \eta. \tag{43}$$

As we shall see, the necessary conditions (30), (31) and (37) simplify considerably as $\varepsilon \to 0$.

5. THE EIGENVALUE PROBLEM

Choi and Horgan (1977) have shown that (18) and (19) have decaying solutions of the form $e^{-\gamma\xi}\,\theta(\eta)$, $\mathscr{R}\gamma > 0$, where θ satisfies the eigenvalue problem

$$L\theta \equiv \varepsilon\theta'''' + \gamma^2\theta'' + \varepsilon\gamma^4\theta = 0, \quad |\eta| < 1, \tag{44}$$

$$\theta(\pm 1) = \theta'(\pm 1) = 0. \tag{45}$$

Further, they have shown that if $0 < \varepsilon < \frac{1}{2}$, then, with

$$2q_{\pm} = \sqrt{\varepsilon^{-1}+2} \pm \sqrt{\varepsilon^{-1}-2}, \tag{46}$$

the eigenfunctions are either even and proportional to

$$\theta^e = \cos q_-\gamma \cos q_+\gamma\eta - \cos q_+\gamma \cos q_-\gamma\eta, \tag{47}$$

where γ satisfies the eigencondition

$$q_- \tan q_-\gamma = q_+ \tan q_+\gamma, \tag{48}$$

or else odd and proportional to

$$\theta^o = \frac{\sin q_-\gamma \sin q_+\gamma\eta - \sin q_+\gamma \sin q_-\gamma\eta}{q_-\gamma \sin q_+\gamma}, \tag{49}$$

where γ satisfies the eigencondition

$$q_- \cot q_-\gamma = q_+ \cot q_+\gamma. \tag{50}$$

Because $\tan \bar{z} = \overline{\tan z}$, the solutions of (48) and (50) occur in conjugate pairs.

Orthogonality

Let $\{\theta_k, \gamma_k\}$ and $\{\theta_l, \gamma_l\}$ denote distinct eigensolutions of (44) and (45). Then, integrating by parts on the left of the identity

$$\int_{-1}^{1} (\theta_l L\theta_k - \theta_k L\theta_l)\,d\eta \equiv 0, \tag{51}$$

we obtain

$$\int_{-1}^{1} \theta_k'(\eta)\theta_l'(\eta)\,d\eta = \varepsilon(\gamma_k^2+\gamma_l^2)\int_{-1}^{1} \theta_k(\eta)\theta_l(\eta)\,d\eta, \quad k \neq l. \tag{52}$$

Likewise, the identities

$$\int_{-1}^{1} (\gamma_l^2\theta_l L\theta_k - \gamma_k^2\theta_k L\theta_l)\,d\eta \equiv 0 \tag{53}$$

and

$$\int_{-1}^{1} (\gamma_l^4\theta_l L\theta_k - \gamma_k^4\theta_k L\theta_l)\,d\eta \equiv 0 \tag{54}$$

lead to the relations

$$\int_{-1}^{1} \theta_k''(\eta)\theta_l''(\eta)\,d\eta = \gamma_k^2\gamma_l^2\int_{-1}^{1} \theta_k(\eta)\theta_l(\eta)\,d\eta, \quad k \neq l \tag{55}$$

and

$$\varepsilon(\gamma_k^2+\gamma_l^2)\int_{-1}^{1} \theta_k''(\eta)\theta_l''(\eta)\,d\eta = \gamma_k^2\gamma_l^2\int_{-1}^{1} \theta_k'(\eta)\theta_l'(\eta)\,d\eta, \quad k \neq l. \tag{56}$$

Equations (52) and (55), due to Grinberg (1953) and Choi and Horgan (1977), respectively, imply (56). As the eigenfunctions and eigenvalues are, in general, complex, variants of these orthogonality relations may be obtained by replacing $\{\theta_k, \gamma_k\}$ by $\{\bar{\theta}_k, \bar{\gamma}_k\}$.

Finally, we note that (55) may be rewritten as the *vector* orthogonality condition

$$\int_{-1}^{1} [\theta_k''(\eta), i\gamma_k^2\theta_k(\eta)][\theta_l''(\eta), i\gamma_l^2\theta_l(\eta)]^T\,d\eta = 0, \quad k \neq l \tag{57}$$

where "T" denotes "transpose".

6. ASYMPTOTIC BEHAVIOR

Our aim now is to examine the asymptotic behavior of the solutions to Cases A–D in the limit as $\varepsilon \to 0$. To this end we first note from (46) that

$$\varepsilon^{-1/2}q_- \equiv Q_-(\varepsilon) = 1+\tfrac{1}{2}\varepsilon^2+O(\varepsilon^4) \quad \text{and} \quad \varepsilon^{1/2}q_+ \equiv Q_+(\varepsilon) = 1-\tfrac{1}{2}\varepsilon^2+O(\varepsilon^4). \tag{58}$$

Thus, the even eigencondition, (48), has the asymptotic form

$$\varepsilon \sin \varepsilon^{1/2}\gamma \cos \varepsilon^{-1/2}\gamma \sim \cos \varepsilon^{1/2}\gamma \sin \varepsilon^{-1/2}\gamma. \tag{59}$$

Here and henceforth, it is understood that the asymptotic approximation symbol, "$\sim$" carries with it the qualifier, "as $\varepsilon \to 0$". In Appendix A, we show that $\mathscr{I}\gamma = O(\varepsilon^{1/2})$. Since $|\sin(x+iy)|$ and $|\cos(x+iy)|$ are each bounded by e^y, it follows that the left side of (59) vanishes as $\varepsilon \to 0$; hence so must the right side. Thus, either

$$\gamma \sim \varepsilon^{1/2}k\pi, \quad k = 1,2,\ldots, \tag{60}$$

or else

$$\gamma \sim \varepsilon^{-1/2}(m-\tfrac{1}{2})\pi, \quad m = 1,2,\ldots. \tag{61}$$

Likewise, the odd eigencondition, (50), has the asymptotic form

$$\varepsilon \cos \varepsilon^{1/2}\gamma \sin \varepsilon^{-1/2}\gamma \sim \sin \varepsilon^{1/2}\gamma \cos \varepsilon^{-1/2}\gamma \tag{62}$$

which may be further simplified to

$$\varepsilon^{1/2} \sin \varepsilon^{-1/2}\gamma \sim \gamma \cos \varepsilon^{-1/2}\gamma \quad \text{if} \quad |\gamma| \quad \text{is bounded.} \tag{63}$$

Because $\mathscr{I}\gamma = O(\varepsilon^{1/2})$ as $\varepsilon \to 0$, the left sides of (62) and (63) vanish; hence so must the right sides. Thus (63) implies

$$\gamma \sim \varepsilon^{1/2} \overset{(0)}{\lambda}{}^{o}_{k}, \quad k = 1, 2, \ldots, \tag{64}$$

where $\overset{(0)}{\lambda}{}^{o}_{k}$ is the kth positive root of

$$\tan \lambda = \lambda. \tag{65}$$

(See Table 4.19 of Abramowitz and Stegun, 1964.) On the other hand, if $\gamma \to \infty$ as $\varepsilon \to 0$, then (62) implies

$$\gamma \sim \varepsilon^{-1/2} m\pi, \quad m = 1, 2, \ldots. \tag{66}$$

A more detailed analysis of the eigenconditions shows that, for the *small* solutions of (48),

$$\varepsilon^{-1/2}\gamma \equiv \lambda^{e}_{k} = \overset{(0)}{\lambda}{}^{e}_{k} + \varepsilon^{2} \overset{(2)}{\lambda}{}^{e}_{k} + O(\varepsilon^{4}), \quad k = 1, 2, \ldots, \tag{67}$$

where

$$\overset{(0)}{\lambda}{}^{e}_{k} = k\pi \quad \text{and} \quad \overset{(2)}{\lambda}{}^{e}_{k} = (3/2)k\pi, \tag{68}$$

whereas for the *large* solutions,

$$\varepsilon^{1/2}\gamma \equiv \Lambda^{e}_{mn} = \overset{(0)}{\Lambda}{}^{e}_{m} - \varepsilon \overset{(1)}{\Lambda}{}^{e}_{mn}(\varepsilon) + O(\varepsilon^{2}), \quad m = 1, 2, \ldots; \quad n = 0, \pm 1, \ldots, \tag{69}$$

where

$$\overset{(0)}{\Lambda}{}^{e}_{m} = (m - \tfrac{1}{2})\pi \tag{70}$$

and $\overset{(1)}{\Lambda}{}^{e}_{mn}$ is the nth root of

$$\Lambda = \cot\left[(m - \tfrac{1}{2})(\pi/\varepsilon) - \Lambda\right], \tag{71}$$

these being ordered in some convenient way. Unless $(m - \frac{1}{2})/\varepsilon$ is an integer, the roots of (71) will be complex (but will occur in conjugate pairs).

Likewise, we find for the small solutions of (50) the more detailed form

$$\varepsilon^{-1/2}\gamma \equiv \lambda^{o}_{k} = \overset{(0)}{\lambda}{}^{o}_{k} + \varepsilon^{2} \overset{(2)}{\lambda}{}^{o}_{k} + O(\varepsilon^{4}), \quad k = 1, 2, \ldots, \tag{72}$$

where $\overset{(0)}{\lambda}{}^{o}_{k}$ is defined by (65) and

$$\overset{(2)}{\lambda}{}^{o}_{k} = -(1/6) \overset{(0)}{\lambda}{}^{o}_{k}, \tag{73}$$

and for the large solutions of (50) the more detailed form

$$\varepsilon^{1/2}\gamma \equiv \Lambda^{\circ}_{mn} = \overset{(0)}{\Lambda}{}^{\circ}_{m} + \varepsilon\overset{(1)}{\Lambda}{}^{\circ}_{mn}(\varepsilon) + O(\varepsilon^2), \quad m = 1, 2, \ldots; \quad n = 0, \pm 1, \ldots, \tag{74}$$

where

$$\overset{(0)}{\Lambda}{}^{\circ}_{m} = m\pi \tag{75}$$

and $\overset{(1)}{\Lambda}{}^{\circ}_{mn}$ is the nth root of

$$\Lambda = \tan(m\pi/\varepsilon + \Lambda), \tag{76}$$

these being ordered in some convenient way. Unless m/ε is an integer, the roots of (76) will be complex (but will occur in conjugate pairs).

Finally, we note that, because $\hat{u}(\eta)$, $\hat{\kappa}(0)$, $\hat{v}(\eta)$, $\hat{f}(\eta)$ and $\hat{f}_{\xi}(\eta)$ are, by assumption, O(1), (31) and (30) imply the *asymptotic necessary conditions*

$$(2/3)\hat{\kappa}(0) \sim -\int_{-1}^{1} \eta\hat{f}(\eta)\,\mathrm{d}\eta \tag{77}$$

and

$$\int_{-1}^{1} \{1, \eta\}\hat{u}(\eta)\,\mathrm{d}\eta \sim 0 \tag{78}$$

which must hold in Cases B and C, D, respectively.

7. THE WIDE BOUNDARY LAYER

By (47), (49) and (58), we may express the eigenfunctions associated with the small eigenvalues (67) in the form

$$\begin{aligned}\theta^{e}_{k} &= \cos[\varepsilon Q_{-}(\varepsilon)\lambda^{e}_{k}(\varepsilon)]\cos[Q_{+}(\varepsilon)\lambda^{e}_{k}(\varepsilon)\eta] - \cos[Q_{+}(\varepsilon)\lambda^{e}_{k}(\varepsilon)]\cos[\varepsilon Q_{-}(\varepsilon)\lambda^{e}_{k}(\varepsilon)\eta]\\ &\equiv \overset{(0)}{\theta}{}^{e}_{k}(\eta) + O(\varepsilon^2),\end{aligned} \tag{79}$$

where

$$\overset{(0)}{\theta}{}^{e}_{k} = \cos k\pi\eta - (-1)^k, \tag{80}$$

and the eigenfunctions associated with the small eigenvalues (72) in the form

$$\begin{aligned}\theta^{o}_{k} &= \frac{\sin[\varepsilon Q_{-}(\varepsilon)\lambda^{o}_{k}(\varepsilon)]\sin[Q_{+}(\varepsilon)\lambda^{o}_{k}(\varepsilon)\eta] - \sin[Q_{+}(\varepsilon)\lambda^{o}_{k}(\varepsilon)]\sin[\varepsilon Q_{-}(\varepsilon)\lambda^{o}_{k}(\varepsilon)\eta]}{\varepsilon Q_{-}(\varepsilon)\lambda^{o}_{k}(\varepsilon)\sin[Q_{+}(\varepsilon)\lambda^{o}_{k}(\varepsilon)]}\\ &\equiv \overset{(0)}{\theta}{}^{o}_{k}(\eta) + O(\varepsilon^2),\end{aligned} \tag{81}$$

where

$$\overset{(0)}{\theta}{}^{o}_{k} = \csc\overset{(0)}{\lambda}{}^{o}_{k}\sin\overset{(0)}{\lambda}{}^{o}_{k}\eta - \eta. \tag{82}$$

By direct calculation, we have, with the aid of (65), the orthonormality condition,

$$\int_{-1}^{1} \overset{(0)}{\theta}{}'_{k}(\eta)\,\overset{(0)}{\theta}{}'_{l}(\eta)\,\mathrm{d}\eta = -\int_{-1}^{1} \overset{(0)}{\theta}_{k}(\eta)\,\overset{(0)}{\theta}{}''_{l}(\eta)\,\mathrm{d}\eta = \delta_{kl}\overset{(0)}{\lambda}{}^{2}_{k}, \tag{83}$$

where δ_{kl} is the Kronecker delta and $\overset{(0)}{\theta}_{k}$ denotes either the even or odd asymptotic eigenfunctions, (80) or (82), respectively. This equation, for $k \neq l$, also follows from (52) and the asymptotic relations (60), (64), (79) and (81). Note that $\overset{(0)}{\theta}_k$ and $\overset{(0)}{\theta}_l$ themselves are *not* orthonormal; rather, by (80) and (82),

$$\int_{-1}^{1} \overset{(0)}{\theta}{}_k^e(\eta)\,\overset{(0)}{\theta}{}_l^e(\eta)\,d\eta = \delta_{kl}+2(-1)^{k+l} \quad \text{and} \quad \int_{-1}^{1} \overset{(0)}{\theta}{}_k^o(\eta)\,\overset{(0)}{\theta}{}_l^o(\eta)\,d\eta = \delta_{kl}+2/3. \tag{84}$$

The completeness of the *asymptotic eigenfunctions* $\overset{(0)}{\theta}{}_k^e$ and $\overset{(0)}{\theta}{}_k^o$ is established in Appendix B.

The eigensolutions $\{\theta_k, \varepsilon^{1/2}\lambda_k\}$ associated with the small eigenvalues defined by (67) and (72) lead to solutions of our basic field equation, (18), even or odd in η, of the form

$$\begin{aligned} W(\alpha,\eta,\varepsilon) &= \sum_1^{\infty} a_k(\varepsilon)\exp[-\lambda_k(\varepsilon)\alpha]\theta_k(\eta,\varepsilon) \\ &= \overset{(0)}{W}(\alpha,\eta)+\varepsilon\overset{(1)}{W}(\alpha,\eta)+O(\varepsilon^2), \end{aligned} \tag{85}$$

where

$$\alpha \equiv \varepsilon^{1/2}\xi \tag{86}$$

is a "slow" axial variable,

$$\overset{(0)}{W} = \sum_1^{\infty} \overset{(0)}{a}_k \exp(-\overset{(0)}{\lambda}_k\alpha)\,\overset{(0)}{\theta}_k(\eta), \tag{87}$$

$$\overset{(1)}{W} = \sum_1^{\infty} \overset{(1)}{a}_k \exp(-\overset{(0)}{\lambda}_k\alpha)\,\overset{(0)}{\theta}_k(\eta), \tag{88}$$

and

$$a_k = \overset{(0)}{a}_k+\varepsilon\overset{(1)}{a}_k+O(\varepsilon^2). \tag{89}$$

We call (85) a *wide* boundary layer because it decays within a region that is wide compared to the width of the strip. Note that $\overset{(0)}{W}$ is precisely the solution we would have obtained had we introduced the change of axial variable (86) into (18), set $\varepsilon = 0$ in the resulting equation, and then sought solutions of the form $e^{-\lambda\alpha}\,\overset{(0)}{\theta}(\eta)$ satisfying the traction-free conditions (19), i.e.

$$\overset{(0)}{W}_{,\alpha\alpha\eta\eta}+\overset{(0)}{W}_{,\eta\eta\eta\eta} = 0, \quad \alpha > 0, \quad |\eta| < 1 \tag{90}$$

$$\overset{(0)}{W}(\alpha,\pm 1) = \overset{(0)}{W}_{,\eta}(\alpha,\pm 1) = 0. \tag{91}$$

A deficiency in $\overset{(0)}{W}$—and a related deficiency of the solution (85)—arises because the partial differential equation it satisfies, (90), contains only second derivatives with respect to the axial coordinate. Thus, the Fourier coefficients $\overset{(0)}{a}_k$ cannot, in general, be chosen so that $\overset{(0)}{W}$ satisfies *two* end conditions at $\alpha = 0$.

8. THE NARROW BOUNDARY LAYER

Let

$$\Theta^e \equiv -\frac{\theta^e}{\cos q_+\gamma} = \cos q_-\gamma\eta-\kappa^e\cos q_+\gamma\eta, \tag{92}$$

where, from (47) and (48),

$$\kappa^{c} = \frac{\cos q_{-}\gamma}{\cos q_{+}\gamma} = \frac{q_{-}\sin q_{-}\gamma}{q_{+}\sin q_{+}\gamma}. \tag{93}$$

Likewise, let

$$\Theta^{o} \equiv -q_{-}\gamma\theta^{o} = \sin q_{-}\eta - \kappa^{o}\sin q_{+}\gamma\eta. \tag{94}$$

where, from (49) and (50),

$$\kappa^{o} = \frac{\sin q_{-}\gamma}{\sin q_{+}\gamma} = \frac{q_{-}\cos q_{-}\gamma}{q_{+}\cos q_{+}\gamma}. \tag{95}$$

Then, by (58), we may express the eigenfunctions associated with the large eigenvalues (69) in the form

$$\begin{aligned}\Theta^{c}_{mn} &= \cos[Q_{-}(\varepsilon)\Lambda^{c}_{mn}(\varepsilon)\eta] - \varepsilon K^{c}_{mn}(\varepsilon)\cos[Q_{+}(\varepsilon)\Lambda^{c}_{mn}(\varepsilon)\tau],\\ &\equiv \Psi^{c}_{mn}(\eta,\varepsilon) - \varepsilon S^{c}_{mn}(\tau,\varepsilon)\\ &\equiv \overset{(0)}{\Psi}{}^{c}_{m}(\eta) + \varepsilon[\overset{(1)}{\Lambda}{}^{c}_{mn}(\varepsilon)\eta\overset{(0)}{\Phi}{}^{c}_{m}(\eta) + \overset{(1)}{S}{}^{c}_{mn}(\tau,\varepsilon)] + O(\varepsilon^{2}),\end{aligned} \tag{96}$$

where

$$K^{c}_{mn} \equiv \frac{Q_{-}(\varepsilon)\sin[Q_{-}(\varepsilon)\Lambda^{c}_{mn}(\varepsilon)]}{Q_{+}(\varepsilon)\sin[\varepsilon^{-1}Q_{+}(\varepsilon)\Lambda^{c}_{mn}(\varepsilon)]}, \tag{97}$$

$$\overset{(0)}{\Psi}{}^{c}_{m} = \cos(m-\tfrac{1}{2})\pi\eta, \tag{98}$$

$$\overset{(0)}{\Phi}{}^{c}_{m} = \sin(m-\tfrac{1}{2})\pi\eta, \tag{99}$$

$$\overset{(1)}{S}{}^{c}_{mn} = (-1)^{m}\frac{\cos[(m-\tfrac{1}{2})\pi - \varepsilon\overset{(1)}{\Lambda}{}^{c}_{mn}(\varepsilon)]\tau}{\sin[(m-\tfrac{1}{2})(\pi/\varepsilon) - \overset{(1)}{\Lambda}{}^{c}_{mn}(\varepsilon)]}, \tag{100}$$

and

$$\tau \equiv \varepsilon^{-1}\eta \tag{101}$$

is a "fast" transverse coordinate.

Likewise, we may express the eigenfunctions associated with the large eigenvalues (74) in the form

$$\begin{aligned}\Theta^{o}_{mn} &= \sin[Q_{-}(\varepsilon)\Lambda^{o}_{mn}(\varepsilon)\eta] - \varepsilon K^{o}_{mn}(\varepsilon)\sin[Q_{+}(\varepsilon)\Lambda^{o}_{mn}(\varepsilon)\tau]\\ &\equiv \Psi^{c}_{mn}(\eta,\varepsilon) - \varepsilon S^{o}_{mn}(\tau,\varepsilon)\\ &\equiv \overset{(0)}{\Psi}{}^{o}_{m}(\eta) + \varepsilon[\overset{(1)}{\Lambda}{}^{o}_{mn}(\varepsilon)\eta\overset{(0)}{\Phi}{}^{o}_{m}(\eta) + \overset{(1)}{S}{}^{o}_{mn}(\tau,\varepsilon)] + O(\varepsilon^{2}),\end{aligned} \tag{102}$$

where

$$K^{o}_{mn} \equiv \frac{Q_{-}(\varepsilon)\cos[Q_{-}(\varepsilon)\Lambda^{o}_{mn}(\varepsilon)]}{Q_{+}(\varepsilon)\cos[\varepsilon^{-1}Q_{+}(\varepsilon)\Lambda^{o}_{mn}(\varepsilon)]}, \tag{103}$$

$$\overset{(0)}{\Psi}{}^{o}_{m} = \sin m\pi\eta, \tag{104}$$

$$\overset{(0)}{\Phi}{}^{o}_{m} = \cos m\pi\eta, \tag{105}$$

and

$$\overset{(1)}{S}{}^{\circ}_{mn} = (-1)^{m+1}\frac{\sin[m\pi+\varepsilon\overset{(1)}{\Lambda}{}^{\circ}_{mn}(\varepsilon)]\tau}{\cos[m\pi/\varepsilon+\overset{(1)}{\Lambda}{}^{\circ}_{mn}(\varepsilon)]}. \tag{106}$$

The eigensolutions $\{\Theta_{mn}, \varepsilon^{-1/2}\Lambda_{mn}\}$ associated with the large eigenvalues (69) and (74) lead to solutions of the basic field equation, (18), even or odd in η, of the form

$$\begin{aligned} N(\beta,\eta,\tau,\varepsilon) &= \sum_{1}^{\infty}\sum_{n} A_{mn}(\varepsilon)\exp[-\Lambda_{mn}(\varepsilon)\beta]\Theta_{mn}(\eta,\tau,\varepsilon) \\ &\equiv \overset{(0)}{N}(\beta,\eta)+\varepsilon[\overset{(1)}{\mathcal{N}}(\beta,\eta,\varepsilon)+\overset{(1)}{S}(\beta,\eta,\tau,\varepsilon)]+\mathrm{O}(\varepsilon^2), \end{aligned} \tag{107}$$

where

$$\beta = \varepsilon^{-1/2}\xi \tag{108}$$

is a "fast" axial variable. In (107),

$$\overset{(0)}{N} \equiv \sum_{1}^{\infty}\overset{(0)}{A}_m\exp(-\overset{(0)}{\Lambda}_m\beta)\overset{(0)}{\Psi}_m(\eta), \tag{109}$$

$$\overset{(1)}{\mathcal{N}} \equiv \sum_{1}^{\infty}\exp(-\overset{(0)}{\Lambda}_m\beta)\{[\overset{(1)}{A}_m-\beta\overset{(1)}{B}_m(\varepsilon)]\overset{(0)}{\Psi}_m(\eta)+\overset{(1)}{B}_m\overset{(0)}{\Phi}_m(\eta)\}, \tag{110}$$

and

$$\overset{(1)}{S} \equiv \sum_{1}^{\infty}\exp(-\overset{(0)}{\Lambda}_m\beta)\overset{(1)}{S}_m(\eta,\tau,\varepsilon), \tag{111}$$

where, with $A_{mn}(\varepsilon) = \overset{(0)}{A}_{mn}+\varepsilon\overset{(1)}{A}_{mn}+\cdots$,

$$\overset{(k)}{A}_m \equiv \sum_n \overset{(k)}{A}_{mn}, \tag{112†}$$

$$\overset{(1)}{B}_m \equiv \sum_n \overset{(0)}{A}_{mn}\overset{(1)}{\Lambda}_{mn}(\varepsilon), \tag{113}$$

and

$$\overset{(1)}{S}_m \equiv \sum_n \overset{(0)}{A}_{mn}\overset{(1)}{S}_{mn}(\eta,\tau,\varepsilon). \tag{114}$$

Using (98)–(100) and (104)–(106), we may separate the η and τ dependence in $\overset{(1)}{S}_m$ as follows:

$$\overset{(1)}{S}_m = \overset{(1)}{C}_m(\eta,\varepsilon)\overset{(0)}{\Psi}_m(\tau)+\overset{(1)}{D}_m(\eta,\varepsilon)\overset{(0)}{\Phi}_m(\tau), \tag{115}$$

where

$$\{\overset{(1)}{C}_m, \overset{(1)}{D}_m\} = (-1)^m\sum_n H_{mn}^{-1}\overset{(0)}{A}_{mn}\{\cos\overset{(1)}{\Lambda}_{mn}(\varepsilon)\eta, \sin\overset{(1)}{\Lambda}_{mn}(\varepsilon)\eta\}, \tag{116}$$

$$H^{e}_{mn} = \sin[(m-\tfrac{1}{2})(\pi/\varepsilon)-\overset{(1)}{\Lambda}{}^{e}_{mn}(\varepsilon)] \quad\text{and}\quad H^{o}_{mn} = -\cos[m\pi/\varepsilon+\overset{(1)}{\Lambda}{}^{o}_{mn}(\varepsilon)]. \tag{117}$$

We call (107) a *narrow* boundary layer because it decays within a region that is narrow compared to the width of the strip and we call $\overset{(1)}{S}$ the (lowest-order) *sinuous* part of N because it varies rapidly with η. Note that, because of the presence of the fast transverse variable, $\tau = \varepsilon^{-1}\eta$,

† It turns out that, to lowest order, we need only determine $\overset{(0)}{A}_m$, never $\overset{(0)}{A}_{mn}$.

$$N_{,\eta}(\beta,\eta,\tau,\varepsilon) \sim \overset{(0)}{N}_{,\eta}(\beta,\eta) + \overset{(1)}{S}_{,\tau}(\beta,\eta,\tau,\varepsilon) \tag{118}$$

and

$$N_{,\eta\eta}(\beta,\eta,\tau,\varepsilon) \sim \varepsilon^{-1}\overset{(1)}{S}_{,\tau\tau}(\beta,\eta,\tau,\varepsilon). \tag{119}$$

Had we introduced the change of axial variable (108) into (18) and then set $\varepsilon = 0$, we would have obtained a partial differential equation with only two η-derivatives, implying a concomitant loss of one of the two traction-free face conditions, (19). Then, had we looked for solutions of the form $e^{-\Lambda\beta}\Theta(\eta)$, and imposed only the first of (19), we would have obtained $\overset{(0)}{N}$, i.e. $\overset{(0)}{N}$ satisfies the partial differential equation and boundary conditions

$$\overset{(0)}{N}_{,\beta\beta\beta\beta} + \overset{(0)}{N}_{,\beta\beta\eta\eta} = 0, \quad \beta > 0, \quad |\eta| < 1 \tag{120}$$

$$\overset{(0)}{N}(\beta, \pm 1) = 0. \tag{121}$$

Note that while $\overset{(0)}{N}$ does not satisfy the second of the traction-free conditions in (19), the combination $\overset{(0)}{N}_{,\eta} + \overset{(1)}{S}_{,\tau}$, which appears in (118), does.

In the next section we discuss the imposition of the edge conditions at $\alpha = 0$. In all cases we find that we may satisfy the boundary conditions *to lowest order* without having to consider the sinuous boundary layer. This seems almost paradoxical because, as we just noted, it is the combination $\overset{(0)}{N}_{,\eta} + \overset{(1)}{S}_{,\tau}$, and not $\overset{(0)}{N}_{,\eta}$ alone, that satisfies the second of the two traction-free edge conditions.

9. EDGE CONDITIONS

Because $\varepsilon^{-1/2}W_{,\xi} = W_{,\alpha} = O(W)$ and $\varepsilon^{1/2}N_{,\xi} = N_{,\beta} = O(N)$, the form of the different pairs of end conditions, (20A)–(20D), suggests that the complete solution of (18) satisfying all the prescribed boundary conditions consists of the linear combination of the wide and narrow boundary layers,

$$f = W(\alpha,\eta,\varepsilon) + \varepsilon N(\beta,\eta,\tau,\varepsilon). \tag{122}$$

It is obvious that if the end data are decomposed into even and odd parts, then, by linearity, solutions for f that are even in η may be treated independently of solutions that are odd in η. With this understanding, we omit the superscripts "e" or "o" in what follows. Furthermore, in what follows, we *proceed formally* and consider only the equations for determining $\overset{(0)}{W}$ and $\overset{(0)}{N}$.

Case A

From (20A), (85), (107) and (122), we have

$$\overset{(0)}{W}(0,\eta) = \hat{f}(\eta) \tag{123}$$

and

$$\overset{(0)}{N}_{,\beta}(0,\eta) = \hat{f}_{\xi}(\eta) - \overset{(0)}{W}_{,\alpha}(0,\eta). \tag{124}$$

The coefficients in (87) follow immediately from (123) and the orthogonality condition, (83), as

$$\overset{(0)}{a}_k = -\int_{-1}^{1} \hat{f}(\eta)\,\overset{(0)}{\theta}{}^{e}_k(\eta)\,\mathrm{d}\eta, \tag{125}$$

while the coefficients in (109) follow from (87), (124), and the orthogonality of $\overset{(0)}{\Psi}{}^{e}_m = \cos(m-\frac{1}{2})\pi\eta$ and $\overset{(0)}{\Psi}{}^{o}_m = \sin m\pi\eta$ as

$$\overset{(0)}{A}_m = -(\overset{(0)}{\Lambda}_m)^{-1}\int_{-1}^{1}\left[\hat{f}_{,\xi}(\eta)+\sum_{1}^{\infty}\overset{(0)}{a}_k\,\overset{(0)}{\lambda}_k\,\overset{(0)}{\theta}_k(\eta)\right]\overset{(0)}{\Psi}_m(\eta)\,\mathrm{d}\eta. \tag{126}$$

Thus, in Case A, $\overset{(0)}{W}$ is determined first and then $\overset{(0)}{N}$.

Note that if $\hat{f}(\eta) = \hat{f}^{o}(\eta)$, i.e. if $\hat{f}$ is an odd function, then, because $\hat{f}^{o}(\eta)$ must satisfy the asymptotic necessary condition (77), we have, by (82) and (87),

$$\sum_{1}^{\infty}\overset{(0)}{a}{}^{o}_k = \hat{\kappa}(0) \quad \text{and hence,} \quad \overset{(0)}{a}_1 = \hat{\kappa}(0)-\sum_{2}^{\infty}\overset{(0)}{a}{}^{o}_k.$$

Thus, from (87),

$$\begin{aligned}\overset{(0)}{W}{}^{o} = {} & \hat{\kappa}(0)\exp(-\overset{(0)}{\lambda}{}^{o}_1\alpha)(\csc\overset{(0)}{\lambda}{}^{o}_1\sin\overset{(0)}{\lambda}{}^{o}_1\eta-\eta)+\sum_{2}^{\infty}\overset{(0)}{a}{}^{o}_k[\exp(-\overset{(0)}{\lambda}{}^{o}_k\alpha)(\csc\overset{(0)}{\lambda}{}^{o}_k\sin\overset{(0)}{\lambda}{}^{o}_k\eta-\eta) \\ & -\exp(-\overset{(0)}{\lambda}{}^{o}_1\alpha)(\csc\overset{(0)}{\lambda}{}^{o}_1\sin\overset{(0)}{\lambda}{}^{o}_1\eta-\eta)].\end{aligned} \tag{127}$$

Case B

From (20B), (85), (107) and (122), we have (123) [and hence (125) and (127)] as before, and, additionally,

$$\overset{(0)}{N}_{,\beta\beta}(0,\eta) = [\hat{v}(\eta)+\tilde{\nu}\hat{f}'(\eta)]'. \tag{128}$$

In this case $\overset{(0)}{W}$ and $\overset{(0)}{N}$ are determined independently with

$$\overset{(0)}{A}_m = (\overset{(0)}{\Lambda}_m)^{-2}\int_{-1}^{1}[\hat{v}(\eta)+\tilde{\nu}\hat{f}'(\eta)]'\,\overset{(0)}{\Psi}_m(\eta)\,\mathrm{d}\eta. \tag{129}$$

Case C

Matters simplify if we use (21) to replace the end condition $(20\mathrm{C})_1$ by

$$\int_{0}^{L} f_{,\eta\eta}(\xi,\eta)\,\mathrm{d}\xi+\tilde{\nu}f_{,\xi}(0,\eta) = -\varepsilon^{-1/2}\hat{u}(\eta). \tag{130}$$

Recalling that $\xi = \varepsilon^{-1/2}\alpha = \varepsilon^{1/2}\beta$ and $\eta = \varepsilon\tau$, we obtain from (122),

$$\int_{0}^{L} W_{,\eta\eta}(\alpha,\eta,\varepsilon)\,\mathrm{d}\alpha+\varepsilon^{2}\int_{0}^{\infty}N_{,\eta\eta}(\beta,\eta,\eta/\varepsilon,\varepsilon)\,\mathrm{d}\beta+\tilde{\nu}\varepsilon[W_{,\alpha}(0,\eta,\varepsilon)+N_{,\beta}(0,\eta,\eta/\varepsilon,\varepsilon)] = -\hat{u}(\eta) \tag{131}$$

or, to lowest order, from (87),

$$\sum_{1}^{\infty} (\overset{(0)}{a}_k / \overset{(0)}{\lambda}_k)\, \overset{(0)}{\theta}{}_k''(\eta) = -\hat{u}(\eta). \tag{132}$$

The orthogonality condition, (83), yields

$$\overset{(0)}{a}_k = (\overset{(0)}{\lambda}_k)^{-1} \int_{-1}^{1} \hat{u}(\eta)\, \overset{(0)}{\theta}_k(\eta)\, d\eta. \tag{133}$$

Because $\int_{-1}^{1} \{1, \eta\}\, \overset{(0)}{\theta}{}_k''(\eta)\, d\eta = 0$, (132) implies that (78), the asymptotic necessary conditions for decay, are satisfied.

The coefficients in (109) are again given by (126) with $\overset{(0)}{a}_k$ now given by (133).

Case D

The coefficients in (87) are again given by (133). To find the coefficients $\overset{(0)}{A}_m$ in (109) we use (28) at $\xi = 0$ as a boundary condition instead of $(20D)_2$. Inserting (122) and noting that $\xi = \varepsilon^{-1/2}\alpha = \varepsilon^{1/2}\beta$ and $\eta = \varepsilon\tau$, we find, to lowest order, that

$$\int_{-1}^{1} \{[(\eta/4)(3-\eta^2) - \tfrac{1}{2}]\, \overset{(0)}{N}_{,\beta\beta}(0,\eta) + (3\tilde{v}/2)\eta\, \overset{(0)}{W}(0,\eta)\}\, d\eta$$

$$+ \int_{-1}^{\eta} \overset{(0)}{N}_{,\beta\beta}(0,\mu)\, d\mu - \tilde{v}\, \overset{(0)}{W}_{,\eta}(0,\eta) = \hat{r}(\eta). \tag{134}$$

Multiplying both sides by $\overset{(0)}{\Psi}{}_m'(\eta)$, integrating from $\eta = -1$ to $\eta = 1$, and noting that $\overset{(0)}{\Psi}_m(\pm 1) = 0$, we obtain, after a further integration by parts,

$$-\int_{-1}^{1} \overset{(0)}{N}_{,\beta\beta}(0,\eta)\overset{(0)}{\Psi}_m(\eta)\, d\eta = \int_{-1}^{1} [\hat{r}(\eta) + \tilde{v}\, \overset{(0)}{W}_{,\eta}(0,\eta)]\, \overset{(0)}{\Psi}{}_m'(\eta)\, d\eta. \tag{135}$$

Substituting (87) and (109) into this expression, we have, by the orthogonality of the $\overset{(0)}{\Psi}_m$s,

$$\overset{(0)}{A}_m = -(\overset{(0)}{\Lambda}_m)^{-2} \int_{-1}^{1} [\hat{r}(\eta) + \tilde{v} \sum_{1}^{\infty} \overset{(0)}{a}_k\, \overset{(0)}{\theta}{}_k'(\eta)]\, \overset{(0)}{\Psi}{}_m'(\eta)\, d\eta. \tag{136}$$

Finally, we examine the asymptotic form of (39), our alternative to the Gregory–Wan necessary conditions for Case D. Since the end conditions on g are that the associated displacements be zero, it follows from (33) that

$$\tilde{u}^i(0,\eta) = -u_\xi^i(0,\eta) = \{0, 0, (1/6)(1+\tilde{v}\varepsilon)\eta(\eta^2 - 3/5)\}. \tag{137}$$

But if $i = T$ or $i = B$, then, from (133), $\overset{(0)}{W} = 0$, so that

$$\tilde{g}^j = O(\varepsilon), \quad \tilde{g}^j_{,\xi} = O(\varepsilon^{1/2}), \quad j = \{T, B\}. \tag{138}$$

Thus, (39) implies the asymptotic conditions (78).

The third necessary condition implied by (39) reduces, asymptotically, to

$$(2/3)\hat{\kappa}(0) \sim \int_{-1}^{1} \tilde{u}^r(\eta)\tilde{g}^F(0,\eta)\, d\eta, \tag{139}$$

which is the replacement for (77) when u rather than f is prescribed at the end of the strip.

Note that $\tilde{g}^{\mathrm{F}}(0,\eta)$ is *easy* to compute to lowest order, which is all we need to apply (139). For by (65), (82), (87) and (122),

$$\tilde{g}^{\mathrm{F}} \sim \sum_{1}^{\infty} \overset{(0)}{a_k^0} \exp(-\overset{(0)}{\lambda_k^0}\alpha)(\csc \overset{(0)}{\lambda_k^0} \sin \overset{(0)}{\lambda_k^0}\eta - \eta) \tag{140}$$

where, by (133) and (137),

$$\begin{aligned} \overset{(0)}{a_k^0} &= (1/6)(\overset{(0)}{\lambda_k^0})^{-1} \int_{-1}^{1} \eta(\eta^2 - 3/5)(\csc \overset{(0)}{\lambda_k^0} \sin \overset{(0)}{\lambda_k^0}\eta - \eta)\,\mathrm{d}\eta \\ &= (2/3)(\overset{(0)}{\lambda_k^0})^{-3}. \end{aligned} \tag{141}$$

10. LOWEST-ORDER STRESSES AND DISPLACEMENTS

From (5), (15), (118), (119) and (122), we have

$$\sigma_x/\sigma_0 = f_{,\eta\eta} \sim \overset{(0)}{W}_{,\eta\eta}(\alpha,\eta) + \underline{\overset{(1)}{S}_{,\tau\tau}(\beta,\eta,\tau,\varepsilon)} \tag{142}$$

$$(E_x/E_y)^{1/4}(\tau/\sigma_0) = -f_{,\xi\eta} \sim -\varepsilon^{1/2}[\overset{(0)}{W}_{,\alpha\eta}(\alpha,\eta) + \overset{(0)}{N}_{,\beta\eta}(\beta,\eta) + \underline{\overset{(1)}{S}_{,\beta\tau}(\beta,\eta,\tau,\varepsilon)]} \tag{143}$$

$$(E_x/E_y)^{1/2}(\sigma_y/\sigma_0) = f_{,\xi\xi} \sim \overset{(0)}{N}_{,\beta\beta}(\beta,\eta) \tag{144}$$

while from (17), (21), (26) and (28) we have

$$(E_x^{1/4}E_y^{1/4}/\sigma_0 H)U = -\int_{\xi}^{\ell} f_{,\eta\eta}(\zeta,\eta)\,\mathrm{d}\zeta - \tilde{\nu} f_{,\xi}(\xi,\eta) \sim -\varepsilon^{-1/2}\int_{\alpha}^{\ell} \overset{(0)}{W}_{,\eta\eta}(\mu,\eta)\,\mathrm{d}\mu \tag{145}$$

$$\begin{aligned} (E_x^{1/2}E_y^{1/2}/\sigma_0 H)V = &-(3/2)(1+\tilde{\nu}\varepsilon)\varepsilon^{-1}\int_{-1}^{1} \eta f(\xi,\eta)\,\mathrm{d}\eta \\ &+ \int_{-1}^{1} \{[(\eta/4)(3-\eta^2) - 1/2] f_{,\xi\xi}(\xi,\eta) + (3\tilde{\nu}/2)\eta f(\xi,\eta)\}\,\mathrm{d}\eta \\ &+ \int_{-1}^{\eta} f_{,\xi\xi}(\xi,\mu)\,\mathrm{d}\mu - \tilde{\nu} f_{,\eta}(\xi,\eta) \sim -(3/2)\varepsilon^{-1}\int_{-1}^{1} \eta \overset{(0)}{W}(\xi,\eta)\,\mathrm{d}\eta. \end{aligned} \tag{146}$$

An analysis of the underlined terms in (142) and (143) awaits. Note, however, that if we are interested in computing only dominant stresses and displacements, then we need consider only the underlined term in (142), and then only within the narrow boundary layer at the left end of the strip.

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APPENDIX A

To examine the behavior of the imaginary part of γ as $\varepsilon \to 0$, we set

$$\gamma = \gamma_R + i\gamma_I. \tag{A1}$$

Because we are looking for decaying solutions of (18), we assume that $\gamma_R > 0$. Further, without loss of generality, we may also assume that $\gamma_I \geq 0$ because any complex roots must occur in conjugate pairs.

Turning first to the even eigencondition, (48), we note that

$$|\tan q\gamma| = \left|\frac{e^{-iq\gamma_R} - e^{-2q\gamma_I} e^{iq\gamma_R}}{e^{-iq\gamma_R} + e^{-2q\gamma_I} e^{iq\gamma_R}}\right|, \tag{A2}$$

where q stands for either q_+ or q_-. Thus, applying the triangle inequality to the numerator and denominator in (A2), we find that if $\gamma_I \neq 0$,

$$\tanh q\gamma_I \leqslant |\tan q\gamma| \leqslant \coth q\gamma_I. \tag{A3}$$

Now suppose that $\varepsilon^{-p}\gamma_I$, $p < \frac{1}{2}$, is bounded away from zero, say $0 < 2m \leqslant \varepsilon^{-p}\gamma_I$. Since $2q_+ > \varepsilon^{-1/2}$ by (46), it follows from (48) and (A3) that

$$|q_-/q_+|\,|\tan q_-\gamma| = |\tan q_+\gamma| \geqslant \tanh q_+\gamma_I \geqslant \tanh \varepsilon^{-(1/2-p)}m \to 1 \quad \text{as} \quad \varepsilon \to 0. \tag{A4}$$

But, by (58), $|q_-/q_+| \sim \varepsilon$. Hence, $|\tan q_-\gamma|$ must grow at least as fast as ε^{-1} as $\varepsilon \to 0$. On the other hand, because $2q_- > \varepsilon^{1/2}$ and $\gamma_I \geqslant 2m\varepsilon^p$, it follows from (A3) that

$$|\tan q_-\gamma| \leqslant \coth q_-\gamma_I \leqslant \coth m\varepsilon^{(p+1/2)} \sim \frac{1}{m\varepsilon^{(p+1/2)}} \quad \text{as} \quad \varepsilon \to 0. \tag{A5}$$

Thus, because $p < \frac{1}{2}$, we have a contradiction. That is, as $\varepsilon \to 0$, *the imaginary part of* γ *is, at most,* $O(\varepsilon^{1/2})$.

We now show that, in fact, $\gamma_I = o(\varepsilon^{1/2})$. To this end, we set

$$\gamma = \gamma_R + i\varepsilon^{1/2}\bar{\gamma}_I, \quad \bar{\gamma}_I = O(1). \tag{A6}$$

It then follows that the real and imaginary parts of (48) have the asymptotic forms

$$\varepsilon \tan \varepsilon^{1/2}\gamma_R \cosh \bar{\gamma}_I \sim \tan \varepsilon^{-1/2}\gamma_R \cosh \bar{\gamma}_I \tag{A7}$$

$$-\varepsilon \tan \varepsilon^{1/2}\gamma_R \sinh \bar{\gamma}_I \sim \cot \varepsilon^{-1/2}\gamma_R \sinh \bar{\gamma}_I. \tag{A8}$$

In (A7), the factor of $\cosh \bar{\gamma}_I$ may be cancelled from both sides; likewise with the factor $\sinh \bar{\gamma}_I$ in (A8), *provided* $\bar{\gamma}_I \neq 0$. However, in this latter case, (A7) and (A8) imply that $\tan^2 \varepsilon^{-1/2}\gamma_R = -1$, which is a contradiction. Hence, $\bar{\gamma}_I = o(1)$. We can show that a similar conclusion holds for the odd eigencondition, (50).

APPENDIX B

Let h be any function such that h'' is piecewise continuous on $[-1, 1]$ and $h(\pm 1) = h'(\pm 1) = 0$, the derivatives at end points being interpreted as left or right handed. Further, let h be decomposd into its even and odd parts by setting

$$h = \tfrac{1}{2}[h(\eta)+h(-\eta)]+\tfrac{1}{2}[h(\eta)-h(-\eta)]$$
$$= h_e(\eta)+h_o(\eta). \tag{B1}$$

The conditions on h imply that h_e'' and h_o'' are each piecewise continuous on $[-1,1]$ and that $h_e(\pm 1) = h_o(\pm 1) = h_e'(\pm 1) = h_o'(\pm 1)$.

To get an eigenfunction representation for h_e, let

$$\overset{(0)}{\psi}{}_0^e = 1/\sqrt{2}, \quad \overset{(0)}{\psi}{}_k^e = \cos k\pi\eta, \quad k = 1,2,\ldots. \tag{B2}$$

Then $\overset{(0)}{\psi}{}_k^e$ satisfies the Sturm–Liouville eigenvalue problem

$$\overset{(0)}{\psi}{}_k^{e''} + (k\pi)^2\overset{(0)}{\psi}{}_k^e = 0, \quad 0 < \eta < 1, \quad \overset{(0)}{\psi}{}_k^{e'}(0) = \overset{(0)}{\psi}{}_k^{e'}(1) = 0. \tag{B3}$$

By a standard theorem (Courant and Hilbert, 1953, p. 360), it follows that if

$$a_k = 2\int_0^1 h_e(\eta)\overset{(0)}{\psi}{}_k^e(\eta)\,d\eta, \tag{B4}$$

are Fourier coefficients, then h_e has the representation

$$h_e = \sum_0^\infty a_k\overset{(0)}{\psi}{}_k^e(\eta), \tag{B5}$$

where the series on the right converges uniformly on $[0,1]$. But

$$h_e(1) = a_0/\sqrt{2}+\sum_1^\infty(-1)^k a_k = 0. \tag{B6}$$

Thus, solving for a_0 and inserting the result into (B5), we have, by (80),

$$h_e(\eta) = \sum_1^\infty a_k[\cos k\pi\eta-(-1)^k] = \sum_1^\infty a_k\,\overset{(0)}{\theta}{}_k^e(\eta), \tag{B7}$$

where, from $(83)_1$ and (B4),

$$a_k = -\int_{-1}^1 h_e(\eta)[\overset{(0)}{\theta}{}_k^e(\eta)]''\,d\eta. \tag{B8}$$

Turning to the representation of h_o, we let

$$\overset{(0)}{\psi}{}_0^o = \sqrt{3/2}\,\eta, \quad \overset{(0)}{\psi}{}_k^o = \csc\overset{(0)}{\lambda}{}_k^o\sin\overset{(0)}{\lambda}{}_k^o\eta, \quad k = 1,2,\ldots, \tag{B9}$$

where $\tan\overset{(0)}{\lambda}{}_k^o = \overset{(0)}{\lambda}{}_k^o$. Then, $\overset{(0)}{\psi}{}_k^o$ satisfies the Sturm–Liouville eigenvalue problem

$$\overset{(0)}{\psi}{}_k^{o''} + (\overset{(0)}{\lambda}{}_k^o)^2\overset{(0)}{\psi}{}_k^o = 0, \quad 0 < \eta < 1, \quad \overset{(0)}{\psi}{}_k^o(0) = 0, \quad \overset{(0)}{\psi}{}_k^{o'}(1) - \overset{(0)}{\psi}{}_k^o(1) = 0. \tag{B10}$$

By the theorem quoted above, h_o has the representation

$$h_o = \sum_0^\infty b_k\overset{(0)}{\psi}{}_k^o(\eta), \tag{B11}$$

where the series converges uniformly on $[0,1]$ and

$$b_k = 2\int_0^1 h_o(\eta)\overset{(0)}{\psi}{}_k^o(\eta)\,d\eta \tag{B12}$$

are Fourier coefficients. But

$$h_o(1) = \sqrt{3/2}\,b_0+\sum_1^\infty b_k = 0. \tag{B13}$$

Hence, solving for b_0 and inserting the result into (B11), we have, by (82),

$$h_o = \sum_1^\infty b_k(\csc\overset{(0)}{\lambda}{}_k^o\sin\overset{(0)}{\lambda}{}_k^o\eta-\eta) = \sum_1^\infty b_k\,\overset{(0)}{\theta}{}_k^o(\eta). \tag{B14}$$

where, from (B11) and (83)$_2$,

$$h_k = -\int_{-1}^{1} h_0(\eta)[\overset{(0)}{\theta}{}_k^o(\eta)]^* \, d\eta. \tag{B15}$$

This establishes the completeness of the asymptotic eigenfunctions $\overset{(0)}{\theta}{}_k^e$ and $\overset{(0)}{\theta}{}_k^o$.